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A comprehensive revealed preference approach to approximate utility maximisation

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Abstract: :H GHYHORS D FRPSUHKHQVLYH UHYHDOHG
VWXG\LQJ DSSUR[LPDWH XWLOLW\ PD[LPLVDWLR
VHOHFWHG IURP D PHQX RQO\ LI LWV XWLOLW\ LV
RI DQ\ RWKHU DYDLODEOH RSWLRQ :H VKRZ WKD
FKRLFHV WKDW YLRODWH WUDQVLWLYLW\ RI LQGL
RI WKH UHYHDOHG VWULFW SUHIHUHQFHV 0RUH
LQGLYLGXDO PD\ IDLO WR PD[LPLVH WKHLU XWLO
UHFRYHU WKHLU WUXH SUHIHUHQFHV IURP WKH R
VDPSOH SUHGLFWLRQV DQG ZHOIDUH FRPSDULVR
PLQLPDO DVVXPSWLRQV RQ WKH HPSLU LFDO IUD
DPRQJVW RWKHUV WR WKH VWXG\ RI FKRLFHV RY
VWDWH FRQWLQJHQW FRQVXPSWLRQ DQG ORWWH

JEL codes: D11, D81, D91

Key words: approximate utility maximisation, revealed preference analysis, nontransitive indifferences, recoverability of preferences, interval order V VDWLVILFLQJ

+QKT`2??2MbBp2 `2p2 H2/ T`272`2

TT`QtBK i2 miBHBi» v K tBKBb

S r2^ .xB2r[b] mHbFB

m;mbi- kykR

#bi` +i

q2 /2p2HQT +QKT`2??2MbBp2 `2p2 H2/ T`272`2M+2 K@i?Q/ 7Q`
K i2 miBHBiv K tBKBb2 iBQM Hi2`M iBp2 Bb b2H2+i2/ 7`QK K2Mm Q
miBHBiv Bb MQi bB;MB}+ MiHv HQr2` i? M i? i Q7 Mv Qi?2` p BH
i? i i?Bb KQ/2H +? ` +i2`Bb2b +?QB+2b i? i pBQH i2 i` MbBiBpBiv
T`2b2`p2 i` MbBiBpBiv Q7 i?2 `2p2 H2/ bi`B+i T`272`2M+2bX JQ`2
i?2 BM/BpB/m H K v 7 BH iQ K tBKBb2 i?2B` miBHBiv 2t +iHv- Bi
i?2B` i`m2 T`272`2M+2b 7`QK i?2 Q#b2`p #H2 / i - K F2 Qmi@Q7
M/ r2H7 `2 +QKT `BbQMbX Pm` `2bmHib `2[mB`2 KBMBK H bbmK
B+ H 7` K2rQ`F M/ `2 TTHB+ #H2- KQM;bi Qi?2`b- iQ i?2 bim/
+QMbmKTiBQM #mM/H2b- bi i2@+QMiBM;2Mi +QMbmKTiBQM- M/

E2vrQ`/b, TT`QtBK i2 miBHBiv K tBKBb iBQM- `2p2 H2/ T`272`2M+
i` MbBiBp2 BM/Bz2`2M+2b- `2+Qp2` #BHBiv Q7 T`272`2M+2b- b i
C1G *H bbB}+ iBQM, .3R- .NR

R AMi`Q/m+iBQM

ai2KKBM; 7`QK 2 `HB2` rQ`F #v _B+?i2` (RNe3) M/ _B+?i2` (RNeN)- MmK2`Qmb /2p2HQ
K2Mib BM i?2 `2p2 H2/ T`272`2M+2 HBi2` im`2 T`QpB/2/ iQQHb 7
miBHBiv K tBKBb iBQM rBi? HBKBi2/ +?QB+2 / i X M BKTQ`i M
i?2 #BHBiv iQ /BbT2Mb2 Q7 Mv M+BHH `v bbmKTiBQMb `2; `/

» U

Q7 T`272`2M+2b M/ 7mHHv 2K#` +2 ??i2`Q;2M2Biv KQM; BM/B
`2p2 H2/ T`272`2M+2 i?2Q`v 2bi #HBb?2b ;2M2` H 7` K2rQ`F 7
+?QB+2- 2biBK iBM; T`272`2M+2b 7`QK Q#b2`p #H2 / i - b r2HH
/B+iBQMb M/ / i @/`Bp2M r2H7 `2 +QKT `BbQMbX h?2b2 K2i?Q
+ iBQM iQ i?2 2KTB`B+ H M HvbBb Q7 +QMbmK2` /2K M/- iBK2
M/ mM+2`i BMiv- M/ KmHiBT2`bQM ?Qmb[R]2?QH/ KQ/2Hb- KQM;
i i?2 b K2 iBK2- bB;MB}+ Mi MmK#2` Q7 2KTB`B+ H bim/B2
BM/BpB/m M HHQ M2b Bbi2Mi 2MQm;? iQ #2 +QM;`m2Mi[k] rBi? @iBHBiv
` HHv- i?Bb TQb2b i?2 7mM/ K2Mi H [m2biBQM r??2i?2` i?2 bi M/
Bb M TT`QT`B i2 /2b+`BTiBQM Q7 ?mK M #2? pBQm`X AM //Bi
TQBMi- r?2M2p2` i?2 / i `2 BM+QMbBbi2Mi rBi? miBHBiv K tBKB
T`272`2M+2 iQQHb `2 MQ HQM;2` TTHB+ #H2- b i?2v +`BiB+
#2BM; ++Q`/ Mi rBi? i?2 +H bbB+ MQiBQM Q7 ` iBQM HBivX
h?2`2 `2 i H2 bi irQ r vb Q7 //`2bbBM; i?Bb Bbbm2X PM2 Bb
miBHBiv K tBKBb iBQM 2MiB`2Hv M/ 2tTHQ`2 Qi?2` KQ/2Hb Q7
KBMBbiB+ Q` biQ+? biB+X am+? ` /B+ H /2T `im`2 7`QK i?2 bi
+ Mi/` r# +FbX 6B`bi Q7 HH- BirQmH/ `2[mB`2 M Qp2`? mH Q7
T `i Q7 i?2 2+QMQKB+ M HvbBb #mBHi `QmM/ /2i2`KBMBbiB+ m
i?2 `B+?M2bb Q7 / i `2[mB`2/ iQ bim/v +?QB+2b Q7 BM/BpB/m H
2Hb Bb T`Q?B#BiBp2 BM K Mv 2KTB`B+ H b2iiBM;b BM r?B+? i?
72r +?QB+2b 7`QK bK HH MmK#2` Q7 K2MmbX

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the data in the approximate sense, any (possibly unobserved) choice from $,'$ induced by such a model would be preferable to any choice from the set $,$, with respect to the true preferences . This captures the idea that, although the decision-maker may fail to maximise their utility, the researcher evaluates their welfare using the true preferences, thus, separating the positive and normative aspects of the choice.[6]

We do not claim that approximate utility maximisation is the ultimate explanation for *any* deviation from the classic notion of rationality. Inevitably, some departures require a qualitatively diferent approach to modelling consumer choice. Rather, the point of this paper is to develop a comprehensive and meaningful framework for studying the canonical model of utility maximisation when the empirical data exhibit inconsistencies with the theory that could be attributed to non-transitive indiferences.

The idea of non-transitive indiferences was introduced to economics by

Organisation of the paper In Section 2 we introduce our setup and the basic notation. Our frst main result (Theorem 1) is presented in Section 3, where we characterise approximate utility maximisation in terms of non-transitive indifferences. We devote Section 4 to a discussion on imperfect discrimination and how this phenomenon can be studied using the toolkit developed in this paper. Theorem 2 is stated in Section 5, where we discuss the problem of eliciting the true preferences of approximate utility maximisers from their observable choices. We present the welfare analysis in Section 6, where we state our fnal main result (Theorem 3). Section 7 is devoted to some direct applications of our method. Specifcally, in Section 7.1, we determine the class of empirical settings in which approximate utility maximisation is indistinguishable from the model of satisfcing à la Simon (1947). In Section 7.2 we extend the results in Polisson et al. (2020) to study approximate utility maximisation within a broad class of models of choice under risk, that includes expectedectede ⁊ e= eķǾÂψ – ɦ und!

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r 2

The frst natural application is the classic consumer demand problem as in

kXk *?QB+2 KQMQiQMB+Biv

Pm` MQiBQM Q7 ` iBQM HBb #BHBiv Bb ` i?2` r2 FX AM T `iB+

elements in the domain. Nevertheless, we require the following basic properties.

Assumption 1 (Partial order). The correspondence $\Phi : X \rightrightarrows X$ satisfies:

(i) For any $x \in X$, we have $x \notin \Phi(x)$.

(ii) For any $x, y \in X$, if $y \in \Phi(x)$ then $\Phi(y) \subseteq \Phi(x)$.

The first condition guarantees that no alternative is objectively superior to itself. The second restriction imposes a form of transitivity on Φ.[13]

3 Revealing non-transitive indifferences

The goal of this section is to provide a revealed preference characterisation of approximate utility maximisation and discuss its relation to non-transitive indifferences. To give a better context for our analysis, we begin by presenting the observable implications of the classic model of the *exact* utility maximisation.

3.1 Classic revealed preference analysis

The proof is straightforward.[14] With the additional notation in place, we return to our initial question. Given a correspondence , under what conditions can we rationalise the dataset O with a -monotone utility maximisation; i.e., when there is a function for which the choice model in (

yielding a contradiction. Theorem 2 in Nishimura et al. (2017) states that, under Assumption 1 and some regularity conditions, this is also a sufficient condition for a dataset to be rationalisable in this sense.[16] Moreover, within the classic demand framework of Afriat, the above restriction coincides with the well-known *generalised axiom of revealed preference* (or GARP) introduced in Varian (1982).

Since transitive indifferences are critical for utility maximisation, the only revealed preference cycles admissible by this model are those induced by the weak relation alone, i.e., where ^{i} $^{i+1}$, for all $= 1$ 1, and ^{n} ^{1}. For any such sequence, each alternative in the cycle must be indifferent to all others. In the following subsection we investigate implications of non-transitive indifferences.

3.2 The main result

Once we relax transitivity of indifferences, it is possible to observe revealed preference cycles along which some alternatives are ordered with . However, as we maintain transitivity of the strict preference, the directly revealed strict preference relation must be *acyclic*. That is, there is no sequence ^{1} ^{2} ^{n} in such that

$$^{1} \quad ^{2} \quad ^{2} \quad ^{3} \quad ^{n\;1} \quad ^{n} \quad \text{and} \quad ^{n} \quad ^{1} \tag{4}$$

This condition excludes any revealed preference cycles that are induced by the revealed strict relation alone. Although acyclicity of remains necessary for the dataset to be rationalisable with utility maximisation, it is no longer sufficient, as it allows for cycles that are generated by the weak and the strict relations jointly.

Before stating our main result, we impose one final assumption.

Assumption 2 (Weak separability). There is a countable set such that 2 $()$ implies $()$ $()$ and 2 $()$, or 2 $()$ and $()$ $()$, for some 2 .

This condition holds trivially whenever is countable, as one can always choose $=$ and set $=$ or $=$, for any 2 satisfying 2 $()$. However, weak separability of is indispensable when considering general spaces.

Theorem 1. *For an arbitrary dataset O and a correspondence : $\rightrightarrows$ satisfying Assumptions 1 and 2, the following statements are equivalent.*

[16] Although our notation differs, this condition is equivalent to *cyclical* $\trianglerighteq$*-consistency* in Nishimura et al. (2017) for the relation (preorder) $\trianglerighteq$ $\{(y;x) \;\; y \; 2 \; \Gamma(x)\} \; [\; \{(x;x) \;\; x \; 2 \; X\}$.

(i)

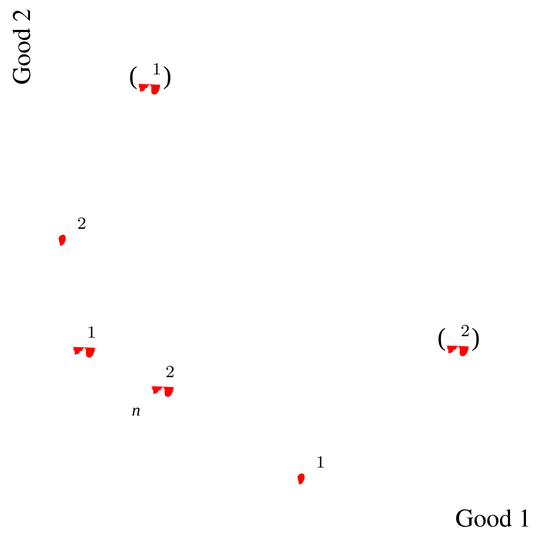

Figure 1: The dataset is *not* approximately rationalisable for a continuous utility u.

explore this connection further, and show that acyclicity of $\backslash$ is necessary and sufficient for a dataset to be rationalisable with an interval order maximisation.

3.3 Continuous approximate utility maximisation

Given the generality of our setup, Theorem 1 does not specify any particular properties of the function that rationalises the data as in (5). Specifically, the utility need not be continuous, even in very well-behaved choice environments.

Consider the dataset $O = \{(\ ^1\ ^1)\ (\ ^2\ ^2)\}$ depicted in Figure 1, where $= \mathbb{R}^2_+$. Suppose that the correspondence is given by $( ) := \{ \ 2 \ : \ \}$. One can easily verify that it satisfies Assumptions 1 and 2.[20] We claim that the dataset is rationalisable with approximate utility maximisation. Since $^2\ 2\ ^1$ and $(^1) \setminus\ ^2 \neq ;$ imply $^1\ \ ^2$ and $^2 \backslash\ ^1$, respectively, the relation $\backslash$ is acyclic. Therefore, by Theorem 1, there is a utility and a positive threshold that rationalise the data as in (5).[21] However, any such function must be discontinuous at ^{2}.

Indeed, since $^2 \backslash\ ^1$ and $^1\ \ ^2$ imply $(^2)\ \ (^1)$ and $(^1)\ \ (^2)\ \ (^2)$, respectively, the two relations hold simultaneously only if $(^2)\ \ 0$. Take any sequence of alternatives $f\ ^n g$ converging to ^{2} such that $^2\ 2\ (\ ^n)$, i.e., $^2\ \ ^n$, for all . Since -monotonicity requires that $(^2)\ \ (\ ^n)\ \ (^2)$, for all , the utility function would be continuous only if $(^2) = 0$, yielding a contradiction.

[20] Clearly, it obeys Assumption 1. By Lemma 4.1 in Peleg (1970), it satisfies Assumption 2.

[21] Since $x^1 R^* x^2 P^* x^1$, this set is *not* rationalisable with an *exact* utility maximisation.

Given the importance of continuity for establishing non-emptiness of the set,

72+i /Bb+`BKBM iBQM bim/B2/ 2ti2MbBp2Hv B MQiH2 QbBv+; QT2v

2KTB`B+ H 2pB/2M+2- BM/BpB/m Hb T2`+2Bp2 /Bz2`2M+2b #2ir

mHmb U2X;X- HB;?i- iQm+?- bQmM/V QMHv B7 i?2v `2 bB;MB}+

r2HH@2bi #HBb?2/ q2#2`@62+?M2` H r biBTmH i2b i? i T2QTH2

i?2 ` iBQ Q7 BMi2MbBiB2b 2t+22/b T `iB+mH`

72`2Mi+2 i Bb bT2+B}+ iQ i?2 biBKmHmbX h?2 b K2 H r TTHB2b

MmK2`QbBiB2b- r?B+? K v #2 KQ`2 2AM2p i?2 i`QK1BQMQ KQ7 i?2 b

iBQM r2 2KTHQv i?Bb B/2 iQ +QMbmK2` +?QB+2- #v HHQrBM;

/Bb+`BKBM i2 KQM; #mM/H2b mMH2bb i?2v `2 bm{+B2MiHv /Bz

1t KTH2 R XX R+ #2 i?2

always coincide (recall Proposition 1). Although [illegible]-monotonicity of approximate utility maximisation requires that [illegible] 2 [illegible]([illegible]) implies ([illegible]) [illegible] ([illegible]), the converse is no longer true.[30] This separation of preferences and choice has a footing in empirical evidence. The aforementioned experiment in Nielsen and Rehbeck (2020) shows a systematic inconsistency between the decision-theoretic rules that individuals consider to be desirable (including first order stochastic dominance) and their actual choices. The authors conclude that we estimate the true preferences of the individual?[33]

Throughout this section we take a dataset O and a correspondence as the premise. Moreover, we assume that O is rationalisable with a -monotone approximate utility maximisation as in (5), for some unobserved utility and threshold . By we denote the directly revealed strict preference relation, defned in Section 3.

It is convenient to refer to the notion of the *revealed strict preference relation* , i.e., the transitive closure of . Formally, we ha to

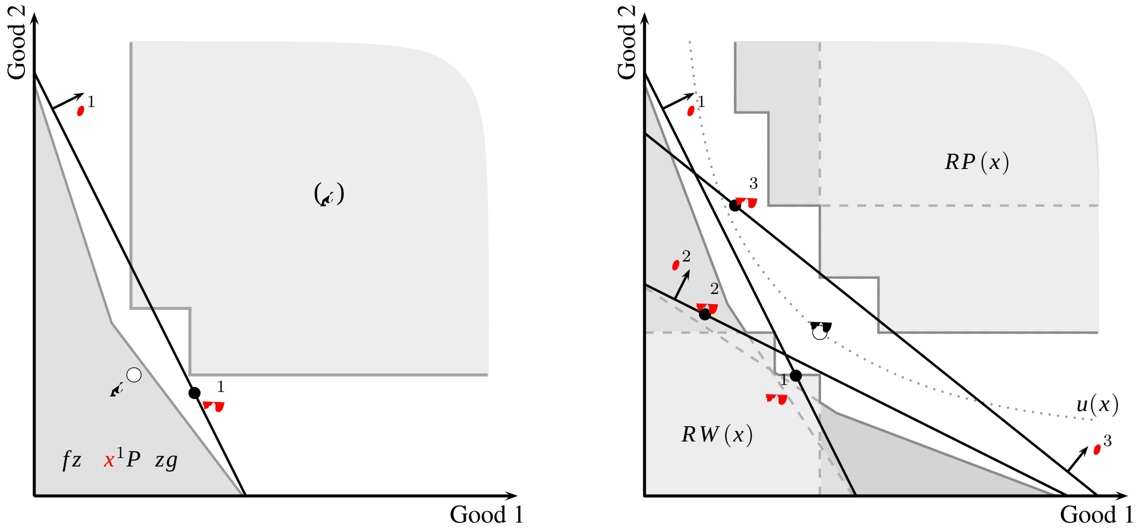

Figure 2: The set of alternatives that are directly revealed strictly inferior to x^1 (left), and the

previous section, we take a dataset O and a correspondence as the premi# /e

The robust comparison over menus is partial and, in general, does not rank any two sets of alternatives. In fact, unlike for the exact utility maximisation, it is possible that two menus $\cdot, \cdot'$ are unordered, even when $\cdot$ is a subset of $\cdot'$. Since choices are not necessarily maximising the utility , there may be alternatives in $\cdot$ that are strictly preferable to some options selected from the set $\cdot'$. However, once $\cdot'$ dominates $\cdot$ in the robust sense, any alternative that would be selected from $\cdot$ is inferior to any alternative chosen from $\cdot'$, even when the individual fails to maximise their utility.

Remark 2. Suppose that Assumptions 1 and 3 are satisfied, and the menu $\cdot$ is compact, for each observation $(\cdot, \cdot) \in O$. Since, by assumption, the set O is rationalisable with a -monotone approximate utility maximisation, Proposition 3 guarantees that the corresponding utility is continuous, without loss. In particular, this suffices for the set $(\cdot)$ to be non-empty, for any compact menu $\cdot$. By combining the arguments supporting Proposition 3 and Theorem 3, one can also prove the following result: *For any compact menus* $\cdot, \cdot' \in A$, *the set* $\cdot'$ *is robustly preferable to* $\cdot$ *if, and only if, for any* continuous *utility and some threshold that rationalise* O *as in (5), the set* $\cdot'$ *is preferred to* $\cdot$ *in the sense defined at the beginning of this section.*

7 Applications

We conclude this paper with a few applications of our main results.

7.1 Satisficing

As it was pointed out in Section 4, approximate utility maximisation can be interpreted in terms of satisficing à la Simon (1947), where the individual selects alternatives that are "good enough" with respect to some criterion. Formally, a choice correspondence $\cdot : A \rightrightarrows$ represents the *satisficing* behaviour if there is a utility $: \to \mathbb{R}$ such that $\cdot \in (\cdot)$ and $(\cdot) \quad (\cdot)$ implies $\cdot \in (\cdot)$, for any $\cdot \in \cdot$ and $\cdot \in A$

One can easily verify that approximate utility maximisation is a special case of satisficing. Indeed, suppose that $(\cdot) = \{ \cdot \in \cdot : (\cdot) + (\cdot) \quad (\cdot) \text{ for all } \cdot \in \cdot \}$, for some utility and threshold function . Since $\cdot \in (\cdot)$ implies $(\cdot) \quad (\ ) \quad (\ )$, for all $\in$

acyclicity of the revealed preference relation is su f cient for the observations to be rationalisable with a -monotone satisfcing behaviour.

The converse is not true. Suppose that = f , g and the correspondence is given by ( ) = f, g, ( ) = f g, and ( ) = ( ) = ;, g which satisfes Assumptions b BM8V U 7Q` bQK2 i?`2b?QXH/aVmM+2+BQM;2M2` H- i?2 +MQ/``2bTQM

`2H iPB QBMM/m+2 M BM}MBi2 MmK#2` Q7 #BM `v +QKT `BbQMb- p2

Bb +QMbBbi2Mi rBi? #Qi? Q7 i?2K K v #2 /B{+mHiX AM i1Bb bm#b

iQ M BKTQ`i Mi +H bb Q7 T`272`2M+2b Qp2` bi i2@+QMiBM;2M

2ti2M/ i?2 K2i?Q/ Q7 H Bb2/ `2bi`B+iBQM U:7_ AMV}#BH2 /QK BM2b HX

UkykV iQ b?Qr i? i rBi?BM #`Q / +H bb Q7 KQ/2Hb +?2M#FBM; 7Q`

P + M #2 `2bi`B+i2/ iQ }MBi2 MmK#2` Q7 +QKT `BbQMbX

amTTQb2 i?2`2 Bb }MBi2=bf2;Q:7;`bi M2/bi?2 T`Q# #BHBiv

2 +? bis i2 S Bb FMQrM iQ i?2 +QMbmK2` M/ i?2 Q#b2`p2`X h?

bmKTiBQM XT=+R2+Bb?2`2 s?2? 2Mi`Q7 i?2 p2+iQ`

For simplicity, we focus on the case where the aggregator is the same across all observations ℓ 2 . Clearly, this is not without loss of generality. For example, when studying the expected utility as in (6), this would require that state probabilities $_s$ remain constant across all observations. Nevertheless, our result can be easily generalised to accommodate a variable aggregator , as we show in the Online supplement. Below we extend Theorem 1 in

As it was pointed out in Section 3, it is not always possible to approximately rationalise a set of observations with a continuous function u. Similarly, Proposition 6 does not guarantee that the Bernoulli function v and, thus, $x \mapsto \big(v(x_1), v(x_2), \ldots, v(x_\ell)\big)$ are continuous. In the Online supplement, we show that whenever the menu B^t is compact, for each observation $t \in T$, and the correspondence satisfies Assumption 3(ii), one can assume that the function v is continuous, without loss of generality.

7.3 A universal measure of departures from rationality

It is a common observation in numerous empirical studies that choices of individuals are not consistent enough to be congruent with the exact utility maximisation. As a result, a significant part of the revealed preference literature is devoted to measures that evaluate how severely the data departs from the classic notion of rationality. Arguably, the most common of them all is the *critical cost-efficiency index* (CCEI, also known as *Afriat's efficiency index*), introduced in Afriat (1973) to evaluate violations of utility maximisation within the standard consumer demand framework.[41]

Throughout this subsection, let $X = \mathbb{R}^\ell_+$ and, for any observation $t \in T$, the corresponding menu be given by $B^t = \{x \in \mathbb{R}^\ell_+ : p^t \cdot x \leq p^t \cdot x^t\}$, for some prices $p^t \in \mathbb{R}^\ell_{++}$. The dataset $O = \{(B^t, x^t) : t \in T\}$ is rationalisable for an *efficiency parameter* $e \in [0, 1]$ (a number) if there is a strictly increasing utility function $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ such that

$$e(p^t \cdot x^t) \geq p^t \cdot x \quad \text{implies} \quad u(x^t) \geq u(x)$$

for all $t \in T$. That is, the observed bundle x^t is preferable to all alternatives that are cheaper than the fraction e of x^t, given prices p^t, for all $t \in T$. Clearly, for $e = 1$, this coincides with the exact utility maximisation. CCEI is equal to the supremum over all efficiency parameters e for which the above condition holds.

Dziewulski (2020) provides a behavioural foundation of this measure. Namely, alt result is established for the general specifcation of the utility function . However, in some applications CCEI is used to measure departures from a specifc formulation of the utility . For example, Cherchye et al. (2017, 2020) apply an analogous measure to a multiperson household model; Polisson et al. (2020) evaluate CCEI for departures from expected utility, rank dependent utility, and disappointment aversion; Cappelen et al. (2021) and Dembo et al. (2021) employ it to estimate deviations from the model of probabilistic sophistication and expected utility maximisation. We apply Proposition 2 to extend the equivalence result to an arbitrary sub-class of utilities.

Proposition 7. *For any dataset O, any strictly increasing utility* $: \mathbb{R}^{\ell}_{+} \to \mathbb{R}$, *and any number*

1 11.9552 Tf 6.652 0 4.553 :y

1 11.9552 Tf 6.652 0 4.555 9.684 =

Our observations allow for a natural extension of CCEI not only to arbitrary utilities , but also to empirical settings beyond the classic demand framework à la Afriat. Given any dataset O with arbitrary menus , one can establish the severity of departures from rationality with the least 1 for which the data can be rationalised as in (5) for the correspondence ( ) := $\{ \; ' \; : \; ' \; \}$. Unlike the original interpretation, our take on CCEI does not depend on the linear specifcation of the budget sets and allows for a meaningful comparison across diferent choice environments. Moreover, given the results in Section 5 and 6, it permits not only to measure departures from rationality, but also to estimate the true preferences of the individual, make out-of sample predictions, and evaluate welfare when the data is not consistent with utility maximisation.

A Appendix

Here we present proofs that were omitted in the main body of the paper. Before stating the argument supporting Theorem 1, it is convenient to prove Proposition 2

Thus, we have (^{t}) ( ), for any such 2 . In particular, for some 2 ,

$$( ) + ( ) = ( ) + ( ) (^t) ( )$$

which contradicts that 2 ( ), and so the correspondence is -monotone.

To prove that rationalises O, take any observation (^{t} ^{t}) and 2 ^{t}. By construction of the threshold , we have ( ) max $\{$ ( ) (^{t}) 0 $\}$ ( ) (^{t}). This suffices for ^{t} to be an element of (^{t}), which concludes the proof.

A.2 Proof of Theorem 1

We prove that statement (i) implies (ii). Given Proposition 2, it suffices to show that there is a utility : ! ℝ such that if 2 ( ) or then ( ) ( ). Before we proceed with our argument, we introduce an auxiliary result.

Lemma A.1. *Let be an irreflexive, transitive binary relation, and be a countable set such that implies either 6 and , or and 6 , for some 2 . Then, there is a function : ! ℝ such that implies ( ) ( ).*

Proof. Take any countable set specified as in the proposition and enumerate its elements so that = f $^k g^1_{k=1}$. For any 2 define the set ( ) := $\{$: $^k g$ and ( ) := $\{$: ^{k} g. One can easily show that implies ( ) ( ) and ( ) ( ), for any 2 . Moreover, at least one of the set inclusions must be strict. Indeed, if and 6 , for some 2 , then ( ) ( ), while 6 and implies ( ) ( ). Define the function : ! ℝ by

$$( ) := \sum_{k \in M(x)} 2^{\;k} \quad \sum_{k \in N(x)} 2^{\;k}$$

which is well-defined and, by our previous observation, consistent with .

Proof. Suppose that $\succ \succ$. By definition, there is some $\in$ such that $\succ^t$ or, equivalently, $(\ ) \cap {}^t \neq \emptyset$. Since $(\ )$ $(\ )$ implies $(\ ) \cap {}^t \neq \emptyset$, we have ^{t}. If $= {}^t$, we are done. Otherwise, we have $\ {}^t$ and ^{t}, which implies . □

The next lemma is an immediate corollary to the previous result.

Lemma A.3. *Under Assumption 1, if $\rhd$ then , for any $\in$.*[43]

Indeed, by definition, we have $\rhd$ if and only if $\in (\ )$. By Assumption 1

Proof. Suppose that ... By Lemma A.4, this holds is three instances. If ..., then ( ) ( ) implies ..., by Lemma A.2. Following the same argument, if $\triangleright$ $'$ ..., for some $'$ $\in$, and ( ) ( ) then $\triangleright$ $'$. Finally, we have $\triangleright$ only if $\in$ ( ) ( ), which implies $\triangleright$. Either way, we obtain . □

Consider the fnal auxiliary result.

Lemma A.7. *Under Assumptions 1*

Let $\wedge$ denote the transitive closure of $[\ \{(\ ,\ )\}$.

Lemma A.10. *Under Assumption 1, the binary relation* $\wedge$ *is irreflexive.*

Proof. Since $\ \not\in\ (\ )$ and $=\ [\ \triangleright\ [\ (\triangleright\ \circ\ )$, we have $\ \not=\ $ by definition of $(\ )$. We consider two cases. If $\ \ $ then $\wedge\ =\ $, which is irreflexive. Otherwise, the relation $\wedge$ fails to be irreflexive only if $\ $ and $\ $, for some $\ \in\ $. However, this implies $\ \ $ which contradicts our initial claim. □

The following lemma shows that $\wedge$ satisfies the separability condition.

Lemma A.11. *Under Assumptions 1 and 2, there is a countable set such that* $'\wedge$ *implies either* $'\wedge\ ''$ *and* $\not\wedge\ ''$, *or* $''\not\wedge\ '$ *and* $''\wedge$, *for some* $\in$

Proof. Take any set specified in Assumption 2 and define $' := [\ \{\ ^t\}_{t\in T}\ [\ \{\ ,\ \}$, which is countable. Suppose that $'\wedge$. If $'\neq$, then either $'=\ =\ $, or $'$ and $\ $. Clearly, the required condition is satisfied for $''=\ $ or $''=\ $.

Alternatively, suppose that $'$. By Lemma A.4, this holds in three instances. If $'$, let $''=\ '\in\ '$. Since $\wedge$ is irreflexive, it must be that $''\not\wedge\ '$ and $''\wedge\ '$. Similarly, if $'\triangleright\ ''$, for some $''\in$, then $''\not\wedge\ '$ and $''\wedge\ '$, where $''\in\ '$.

Finally, suppose that $'\triangleright$. By Assumption 2, either (i) $(\ ')\ \ (\ '')$ and $''\in\ (\ )$, or (ii) $'\in\ (\ '')$ and $(\ '')\ \ (\ )$, for some $''\in$. Whenever (i) is true, then $''\triangleright$, and so $''\wedge$. Towards contradiction, suppose that $''\wedge\ '$. If $''\ \ '$, then $(\ ')\ \ (\ '')$ implies $''\ \ ''$ (by Lemma A.6), yielding a contradiction. Similarly, if $''\ \ $ and $\ \ '$, then $(\ ')\ \ (\ '')$ implies $\ \ ''\ \ $ contradicting that $\ \not=\ $. Thus, we have $''\not\wedge\ '$ and $''\wedge$. Analogously, if (ii) holds, then $'\wedge\ ''$ and $\not\wedge\ ''$, for some $''\in$. □

By Lemmas A.10, A.11, and A.1, there is utility $:\ \to\mathbb{R}$ such that $'\sim$ implies $(\ ')\ \ (\ )$. Therefore, both $'\in\ (\ )$ and $'$ imply $(\ ')\ \ (\ )$, as well as $(\ )\ \ (\ )$. The rest follows from Proposition 2.

A.5 Proof of Proposition 4

Denote $\mathcal{O} = O\ [\ \{(\ ,\ )\}$ and let $\sim\ \ \sim$ be the revealed relations induced by $\mathcal{O}$. In particular, we have $\ \ \sim$. Clearly, the set $\mathcal{O}$ is rationalisable only if $\ \in\ (\ )$. Otherwise, $(\ )\ \backslash\ \ \neq\ ;$ would imply $\ \sim\ $, while $\ \ $ and $(\ )\ \backslash\ \ \neq\ ;$ would imply $\ \sim\ $. Either way, this would contradict that the relation $\sim$ is irreflexive.

We prove the converse by contradiction. Suppose that ℓ 2 ,

03041004G6406d 0032007002d>-333>0065006401 1c006400<00454>69>[TJ /F12 7004d00260001c004 Td[(x)]TJ4[(10790090406406d(

-monotone approximate utility maximisation as in (5).

A.8 Proof of Proposition 7

To show that (i) implies (ii), take any 1 . Following (i), there is some such that 1 1 , and (t t) t implies (t) ( ). By monotonicity of , this guarantees that (t t) t only if (t) ( ). Note that, ( ) \ t ≠ ;, and so t \ , if and only if t t t ( ). Since 1 , this suffices for t \ to imply (t) ( ). Moreover, monotonicity of implies ( ) ( ), for any 1. By Proposition 2, the data is rationalisable as in (5) for the utility .

To show the converse, take any . By (ii), there is some number such that 1 . By the argument above and Proposition 2, we know that t t t ( ) implies (t) ( ). Since 1 , this suffices for (i) to hold.

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K_O ° O}§J ¯ u Nu (1973): “Afriat and revealed preference theory,”

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T ` 2 7 2 ` 2 M + 2 T T ` Q + ? i Q T T ` Q t B K
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S r 2 ^ . x B 2 r m H b F B [»]

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h ? B b b m T T H 2 K 2 M i + Q M i B M b / / B i B Q M H H Q i ` Q b 2 ` p i B Q M b i ? i b ? Q H / i 2 / i Q
M Q i 2 b b ? Q m H / # 2 ` 2 / B M + Q M D m M + i B Q M r B i ? i ? 2 K B M T T 2 ` X

> 2 ` 2 r 2 B M + H m / 2 ` 2 b m H i b i ? i + Q K T H 2 K 2 M i i ? 2 } M / B M ; b T ` 2 b
A M a 2 + i B Q M "X R Q 2 M / B b + m b b M H i 2 ` M i B p 2 - K Q ` 2 + Q M b i ` m + i B p 2 i F 2 Q
H B M 2 ` T ` Q ; ` K K B M ; K 2 i ? Q / b X a T 2 + B } + H H v - i ? B b H H Q r b m b i Q
m i B H B i v 7 m M + i B Q M Q i i 2 b i # H 2 B M + 2 ` i B M + ? Q B + 2 "2 X k p B ` Q M K
r 2 2 t T H Q ` 2 i ? 2 ` 2 H i B Q M # 2 i r 2 2 M T T ` Q t B K i 2 m i B H B i v K t B K B
6 B M H H v - B M a 2 + i B Q M "X j r 2 + B i 2 T ` Q Q 7 b Q 7 i ? 2 ` 2 b m H i b T ` 2 b 2 M i 2 / B M a 2 +
T T 2 ` - ` 2 ; ` / B M ; + ? Q B + 2 Q p 2 ` b i i 2 @ + Q M i B M ; 2 M i + Q M b m K T i B Q
h ? ` Q m ; ? Q m i i ? B b b m T T H 2 K 2 M i r 2 2 K T H Q v i ? 2 M Q i i B Q M B M i `
A M Q ` / 2 ` i Q F 2 2 T Q m ` 2 t T Q b B i B Q M + Q K T + i - r 2 Q K B i i ? 2 H Q M ; 2 ` T ` Q Q 7 b Q 7 b Q K 2 Q
M H B b b 7 2 ` 2 ` 2 B b u M / i ? 2 b ? Q H / 7 m M + i B Q M H B b 2 i ? 2 Q # b 2 ` p
B M i ? 2 b 2 M b 2 b T 2 + B } 2 / B M h ? 2 Q ` 2 K R X ; B p 2 M + Q ` ` 2 b T Q M / 2 M +

[»] . 2 T ` i K 2 M i Q 7 1 + Q M Q K B + b - l M B p 2 ` b B i v Q 7 a m b b 2 t - C m # B H 2 2 " m B H / B M
l M B i 2 / E B M ; / Q K X S X E X . x B 2 r m H b F B ! b m b b 2 t X + X m F

"XRXR *QMbi`m+iBp2 ` iBQM HBb iBQM

:Bp2M Qm` /Bb+mbbBQM BM a2+iBQM j Q7 i?2 K BM T T2`- Bi Bb
Q#b2`p i BQMb iBQM HBb #H2 rBi? TT`QtBK i2 miBHBiv K tBKB
bTQM/BM; /B`2+iHv `2p2 H2/ bi`B+iHv T`27@2`2M+2 h`2Hb iBQM#b2`p iBQ
/B`2+iHv 7`QK i?2 /2}MBiBQM Q7 i?2 `2H iBQM M/ Bb BM/2T2M/2
i?Bb bm#b2+iBQM r2 T`QpB/2 +QMbi`m+iBp2 `;mK2Mi bmTTQ`i
miBHBiv/ i?`2b?QH/ iBQM HBb2 i?2 / i BM i?Bb b2Mb2X

q2 #2;BM Qm` +QMbi`m+iBQM #v /2}MBM; i?2 7mM+iBQM

$$g^t(x) := \begin{cases} f^t(x) & \text{Bf } f^t(x) \quad 0; \\ f^t(x) + & \text{Qi?2`rBb2c} \end{cases} \qquad \text{U"XRV}$$

7Q` bQK2 r?2`2 Bb i?2 r2HH@/2}M2/ M/ bi`B+iHv BM+`2 bBM; 7
b2Mib i?2 K2 MQm`t 2HTHX h?mb- i?2 7mM+iBQM g^t Bb r2HH@/2}M2/ M/ bi
BM+`2 bBM;X JQ`2Qp2`- r2 ? p2 / Q M2 HA $A^t$ B 7Q`t 2HTHX .2}M2
7mM+iBQM R $h^t(x) := \inf \; g^t(y) : y \; 2 \; (x)$ - 7Q`t 2HTHX

G2KK "XRX 7Q` t 2HT- r2 ? p2 $h^t(x)$ 0 B7- M/ QMHv B7 $x \; 2 \; A^t$X

S`QQ7X A7 (x) \ $A^t$ 6= ; i?2M $f^t(y) = g^t(y)$ $h^t(x)$X hQ b?Qr i?2 +QMp2`b2
bmTTQb2 $h^t(x)$ i 0 M/ (x) \ A^t = ; X AM T `iB+mH `- 7Q` M? p2
$g^t(y) = f^t(y) +$ > X h?Bb BK TH B2b $h^t(x)$ > 0- vB2H/BM; +QMi` /B+iBQMX

Ai Bb 2 bv iQ b?Qr i? i i?2 `2p2 H2/ T`27@2`2M+2 `2H iBQM M/ QMHv B7- 7Q
C= (a; b); (b; c); : : : ; (z; a) BM T- r2 ? p2 $(x^s) \setminus A^t = ;$ - 7Q` b(Qs) 2 CX "v
i?2 H2KK #Qp2- i?Bb BK TH B2b $h^t(x^s) > 0$ 7Q` bQK2 b(Qs) 2 CX AM i?2 7QHHQrBM
`2bmHi r2 b?Qr i? i i?Bb bm{+2b iQ bQHp2 T `iB+mH ` HBM2 ` b

G2KK "XkX 2 `2H iBQM +v+HB+ QMHv B7 i?2 `2H iBQM B# 2Hv
TQbBiBp2 (MmK#2` b)- i?2 7Q`t 2HT- ? i < i $^t + ^th^t(x^s)$ - 7Q`t; s 2HT X

h?2 bvbi2K Q7 BM2[m HBiB2b T`2b2Mi2/ BM i?Bb H2KK Bb p
7`B i BM2[m HBiB2b- Bi `2[mB`2b 7Q` HH i?2 BM2[m HBiB2b i
Bib2H7 Bb M HQ;Qmb iQ G2KK kXR BM M #2 T`Qp2M mbBM; M
K2Mi # b2/ QM i?2 QM2 BM 6Q`i2H Bii- Q`2 7Q`2 bi iBM; i?2 K BM `
Q7 i?Bb b2+iBQM- r2 BMi`Q/m+2 QM2 }M H H2KK X

G 2 K K " X j X M / 2 ` b b m K T X R B y M 2 ( x ) i ? 2 h M t ( y ) > h t ( x ) - 7 Q ` t 2 M T X

S ` Q Q ' 7 X K Q M Q i Q M g B + M i / v / 2 Q } 7 M B i B t Q M ? Q ` 7 2 B b z b B Q M K 2 ? 2 + H Q (b y) m ` 2 Q 7

b m + ? i h ? (y) g t (z) X 6 Q H H Q r B M ; b X R i K ? 2 i ` B 2 Q B M (x) b i B b 7 z 0 - B z M X

a B M g t + 2 b b i ` B + i H v B M + ` 2 h t (y) M ; g t (z) > g # (z 0) B M h t (x) X

S`QQa7BXM+l2 iBb}2b bbm"X RhB?Q2M "XR;m ` Mi22b i? i i?2 /Bibb2i ` iBQM HBb #H2 rBi? i?2 u(y) i= mHiB v t 7 mM(y)i BQM - M/ i?`2b?@ QHX "v bi`B+i KQM Qfi Q M?2+7BmM+ iBQM h(y) = f t(y) - B t(y) < 0- Mh t(y) = f t(y) + Qi?2`rBb2- 7Q` 0 X QKH2 `Hv- Bi Bb mTT2` b2KB@+QMiBM T `iB+mH `- i?2 !7mM+iBQM Bb mTT2` b2KB@+QMiBMmQmb- 7Q` M/ bi`B+iHv TQbBiBp2 MmK#2` aBM+2 i?2 T`2b2`p2b mTT2` b2KB@+QMiBMmBiv- i?2 7mM+iBQM @+QMiBMmQmbX □

_2+ HH i?2 2t KTH2 BM 6B;m`2 R BM i?2 K BM T T2`X b r2 T /B`2+iHv `2p2 H2/ bi`B+i T`272`2M+2 `2H iBQM BM/m+2/ #v i?2 b HBM2 `#m/;2i b2ib + M #2 `2T`2b2Mi2/ rBi? bi`B+iHv f M+`2 bB #v i?2 #Qp2 +Q`QHH `v- i?Bb T `iB+mH ` / i b2i + M #2 TT`QtBK mTT2` b2KB@+QMiBMmQmb mi BHBiv 7mM+iBQM Hv X

"XRXk GBKBib iQ i2bi #BHBiv

S`QTQbBiBQM j Q7 i?2 K BM T T2` bT2+B}2b +QMiBiBQMb- mM TT`QtBK i2Hv ` iBQM HBb2b i?2 / i Bb +QMiBMmQmb- rBi?Qm bm+? 2MpB`QMK2MiQM i?2 m"X2Hv ?2b+QMbi`m+iBQM Q7 i?2 mi i?`2b?QH h?2Q"XRH Qr mb iQ 7m`i?2` BMp2biB; i2 T`QT2`iB2b M/ B/2MiB7v +?QB+2 QM p7B`QMK2Mi b2v `2 MQi 7 HbB} #H2X h?`Qm;?Qmi i?Bb bm#b2+iBQM- r2 M/ F2Q i?2 bTQb2/ M22 T`2KBb2X AM //BiBQM- r2 bbmK2 i? i i?2 b2i Q7 Q#b2`p iBQM Bb #H2X q2 #2;BM Qm` /Bb+mbbBQM #v T`2b2MiBM; bm{+B2Mi + + M #2 2tTH BM2/ rBi? +QMiBMmBiv/mQmb 7mM+iBQMb

bbmKTiBQMh"XkX r2` #Q+m`2bTQM/2M+2X - ;Bp2M #v

$$@_{\#}(x) := \left\{ y \in (x) : z < y \text{ BKTHB2b } z \in (x);\ 7Q` z \in X \right\};$$

Bb r2HH@/2}M2/- +QKT +i@p HHm2/- M/ +QMiBMmQmbX[e]

_2+ HH i?2 +Q``2bTQM/2M+2 :X > xgX AM i?Bb + b2- i?2 HQr2` #
$@_{\#}(x)$ Bb 2KTiv- 7Q` -HH B+? pBQH i2b i?2 #Qp2[d] bbmKTiBQMX

[e] a22 .2}MBiBQM RdX 9 M/ AMi`Q/m+iBQM /2}MBiBQM Q7 +QMiBMmQmb TQM/2M+2X
[d] h?2 b K2 TTHB2b iQ i?2 +Q``2bTQM/2M+2 :X

S`QTQbBiBQlMM/"2X`RXbmKT"iBRQMM/"bX+B7 i?2 7mftM`+2iBT`Q2M2MiBM; i?2

K2MAtmBb +QMiBM mQ2mTb– i?Q2M H?H2 /Oi Bbb2i TT`QtBK i2Hv ` iBQM HI

7Q` +QMiBM mQ mMb/ m+iQHBBBwM mQmb ri?î2Q?mOHH/Qbb Q7 ;2M2` HBi

S`QQ.72)XM2 7mMgt+iBXQ!MR bgt(x

S`QQ7?2 }XM2 7mMg+iBQMt M **b** BM a2+iBQM "XR-Q7 Q`t 2HTHX 6B`bi- r2 b?Qr i?2 `2@

bmHi Qmib B/2 i?2 #` +F2Bi **b** XB M?2 2M 2bpB2xM ; -x7 Q`y 2M (x

S ` Q Q 7 X

7Q` i?Bb +Q``2bTQM/2M+2- b b?QrM BM a2+iBQM j Q7 i?2 K B

S`QTQbBiBQM "Xj- QM2 + M ` iBQM HBb2 bm+? / i b2ib rBi? [m bB+QM

S`QTQbBiBQM "Xj- "X9 + M #2 TTHB2/ /B`2+iHv 7Q` i?2 UR N d V2`bBQM Q7

URNdVj- M/ `B URNd3VX aBM+2 i?2 Q`B;BM H 7` K2rQ`F bbmK2b i? i

+ M #2 `2T`2b2Mi2/ rBi? (y) = p(y) +M- 7Q` bQK2 p : B + BM+QK2

m - 7Q`t2HH- i?2 `2[mB`2K2Mib Q7 i?2 irQ `2bmHib `2 b iBb}2/X

"Xk _2H iBQM iQ BMi2`p H Q`/2`b

6B`bi- /2}M2 #BMP v b2xH yiBQ7M y M z 6 x- 7Q` bzQ2KX2X

6QHHQrBM; G2KK 2 bXF2 `BMU2yiyVH XBb bvKK2i`B+ M/ M2; iBp2Hv i

iBp2 XBp2M i? ii?2 +Q``2b TQ@MK/Q M+2 Q M2- Bi K2m (y) #2Ki?HiB2b

x y X Pi?2`rBb2- r2 rQm2 +f x;y p2 +QMi` /B+iBQ M b@KiQ M Q iQ M2X

q2 +H BKxP? yiBKTHB y2X h F2 Mv Q#b(2; x) 2KBQ MM z2 (y) \ A X

*H2 `Hv- Bi Kmbi #y2 iM z 6 x - r?B+? BKTHyHB2b

hQ b?Qr P? Bb +v+HB+- i F2 z; z; : : : ; z 2BM+2m +? iz P iz -

7Q` i = 1; : : : ; (n 1) X h?mb- ;Bp2M i?2 Q#b2`p iBQz P z #QQ2- r2 Q#i

2[mBp H2tM z P z - 7Q`i = 1; : : : ; (n 1) U#v bvKK2P V X 7v M2; iBp2

i` MbBiBp iB iQK7mbi #2 i2 P z - M/ h Q z P z X □

h?Bb `2bmHi +Q K2KH2#bRM b i? i +? ` +i2`Bb2b +?QB+2 +Q``2b

;2M2` i2/ #v M BMi2`p H Q`/2` K tBKBb iBQM mM/2` i?2 bbmK

Q#b2`p2bMi?2`2 (A) 7Q` HHQbbB#H2 A K2 X mA M +QMi` bi- r2 bbmK2 i

i?2 / i `2 BM+QKTH2i2X Pm` `2bmHi `2[mB`2bX M/Q i bb2m KTiBQM

Q` i?2 +Q``2bT QM /2M+2 B+mH `- bBM+2 r2 HHQr 7XQ i Q?#2+QMbm

mM+QmMi #H2- i?2 B Ki2v` pM Q iQ`/p2 `2T`2b2Mi 6 B b?#MR N W X M

JQ`2Qp2`- i?2`2 Bb MQ /B`2+i `2H iBQM #2 P2 2M ii?2 `2Mp2 Hp2 HT`

Q`/2`

bi`B+iHv BM+`2 bBM;- r2 ? p2

"XjXk _2H i2/ `2bmHib

*QMiBMmBiv bi- r2 //`2bb i?2 [m2biBQM Q7 +QMiBMmBiv Q7 i?2
amTTQb2 i? i i A^t 2 Kb2 +QKT +i 7Q` H /Hi?2 +Q``2bTQMbBM;
bbmKTiBQM jUBBVX q2 +H BK i? i i?Bb bm{+2b 7Q` i?2 /"2MMQ
S`QTQbBiBQM e iQ #2 +QMiBMmQmb- rBi?Qmi HQbb Q7 ;2M2` H
AM/22/- BM bm+? + b2- i?2 (A^t) Bb 2 +QKT +i- Bb 2 7Q` HH Bb Bib
/QrMr `/ +QKT`2?2 M(A^t) X2 a?BXM+2b }MBi2- i?2`2 Bb +HQb2/ M2B;
V Q7 (A^t) bm+? i? `\i _`(A^t) = X`\ VX .2M2 i2 = V[ y2 R`+ :y x^t -
r?B+? Bb +QKT +i M(A^t) QM iBM BMBMi2`BQ`X JQ`2Qp2`- 7Q` Mv bi`
7mM+iBQM 2+B}2/ b BM bi i2K2Mi UBBV Q7FS`QTQbBiBQM e- r2 ?

2tT2+i2/ miBHBiv 7Q`KmH iBQM r?2M [mHHHBiv iT2`Q# #H2- BX2X-

$u(x) = \sum_{s=1}^{`} (1=`)v(x_s)$ - 7Q` bQK2 "2`MQmHHB 7mM+iBQM

q?2M2p2` /OiBbi TT`QtBK i2Hv ` iBQM HBb #H2 r-BQ?M2bvKK2

rQmH/ 2tT2+i i?2 +Q``2bTQM/BM; iQ` #21bQKK 7mM+iBQM HHX h?

i?2 ;2Mi b?QmH/ #2 2[m HHv BKT`2bb2/ rBi? B/iBT;2`K mXMiBQM

h?Bb Bb BM/22/ i`m2- rBi?Qmi HQbb Q7 ;2M2` HBivX

S`QTQbBiBQM kX b2i $x$ i y :y2 (x) - 7Q` x2vX M/ Mv T2`Km@

i iBQMX A7 i?2 /OiBbi TT`QtBK i2Hv ` iBQM HBb #H2 u 7Q` / bvKK2

bQK2 i?`2b-?iQHM i?2 7mM+iBQM K2i`B+- rBi?Qmi HQbb Q7 ;2M2` H

S`QQ7X TQb2 i? i i?2 /Bbi i2iBQM HBb #H2 rBi? bvKK2i`B+ miBH

i?`2b?QH/M/ /2}M2 $(y) := \max\ ^0(y)$ : 7Q` bQK2 r?B+? Bb r2HH@/2}M2/

bvKK2i`B+X q2 +H; BKTT?QtBK i2Hv `OiBQM HBb#H2 b?Qr i? i i?2

KQ/2H@BbKQMQiQM2X 2h(F2X Mv bbmKTiBQM -2r(x?) X 2a BuM +2

` iBQM HBb2 i?2/ i -i?2`2 Bb bQK2 TU2)iK (y) = BuQM $^0(y)$ >

$u(x) = u(x)$ X hQ b?Qr i? i i?2 KQ/2H ` iBQM HBb2 i?2/ A X i F2 Mv

h?2 $u(x^t)$ $u(y)$ $^0(y)$ $u(y)$ (y) X □

h?2 //BiBQM H `2bi`B+iBQM QMBiK2 rQ2b2 r2HH7Q`K+2Q7 bvKK2

QM i?2 KQMQiQMB+Biv Q7 +?QB+2X *H2(x) H=v-yi2X :yQM/KiBQM ?

aBKBH `Hv-Bi TTHB2b iQ 1 t_s.299 Td[(2.26J/F12 7.9701 Tf 5.633 4.333.00S Q.4d[

Chateauneuf, A.